Topology Vol. 10. pp. 237–256. Pergamon Press, 1971. Printed in Great Britain.

ON CHARACTERISTIC NUMBERS OF ALMOST COMPLEX MANIFOLDS WITH FRAMED BOUNDARIES

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(*Received* 9 *December* 1969)

THE OBJECTIVE of this paper is to prove the analog of the results of Stong [13] and Hattori [8] for almost complex manifolds with framed boundaries. More precisely we will demonstrate:

THEOREM. *All odd primary relations among the integral cohomology characteristic numbers of compact almost complex manifolds with compatable framed boundaries are given by the integrality of their K-theory characteristic numbers.*

If we denote by **MU** the unitary Thom spectrum [10] [14] and S the sphere spectrum there is then a cofibration sequence of spectra

$$\mathrm{S} \to \mathbf{MU} \to \mathbf{MU/S}$$

defining a spectrum **MU/S**. The spectrum **MU/S** has a bordism interpertation [6] in terms of compact weakly complex manifolds with compatably framed boundaries. Let us denote by **BU** the spectrum [4] representing $\mathbb{Z}$-graded **K**-theory. Then smashing with the unit of the ring spectrum **BU** provides a morphism of spectra

$$u: \mathbf{MU/S} \to \mathbf{BU} \wedge \mathbf{MU/S}.$$

An equivalent formulation of the above theorem is:

THEOREM. *The map of spectra*

$$u: \mathbf{MU/S} \to \mathbf{BU} \wedge \mathbf{MU/S}$$

induces a split monomorphism

$$\frac{\Omega_*^{U,fr}}{\text{Torsion}} \otimes \mathbb{Z}[1/2] \to \mathbf{K}_*(\mathbf{MU/S}) \otimes \mathbb{Z}[1/2]$$

of $\mathbb{Z}[1/2]$*-modules, where* $\mathbb{Z}[1/2]$ *denotes the subring of the rational numbers with denominators a power of two.*

As an application of our proof of the above theorem we will reobtain the odd primary part of the results of Adams [3] concerning the image of the homomorphism

$$e_{\mathbf{C}}: \pi_{2n-1}^S \to \mathbf{Q}/\mathbb{Z}$$

thereby reobtaining with the aid of [1] the result of [3] that the odd primary part of Image J is a direct summand in $\pi_*{}^S$.

§1. PRELIMINARIES ON K-THEORY CHARACTERISTIC NUMBERS

We will be employing **K**-theory characteristic numbers for (U, fr)-manifolds. The standard reference for background material on **K**-theory characteristic numbers is [14; pp. 117–130].

Let us recall that the cofibration of spectra

$$\mathbf{S} \to \mathbf{MU} \to \mathbf{MU/S}$$

leads to the exact sequence

$$0 \leftarrow \mathbf{K}^*(\mathbf{S}) \leftarrow \mathbf{K}^*(\mathbf{MU}) \xleftarrow{\Psi^*} \mathbf{K}^*(\mathbf{MU/S}) \leftarrow 0$$

where

$$\operatorname{Im} \Psi^* = \left\{ \begin{array}{l} \lambda\mathfrak{I} : \mathfrak{I} \in \mathbf{K}^*(MU) \text{ is the Thom} \\ \quad \text{class and } \lambda \neq a\,1 \in \mathbf{K}^*(BU) \\ \quad a \in \mathbb{Z}. \end{array} \right\}$$

Thus the **K**-theory characteristic numbers of (U, fr)-manifolds are

$$\gamma^{\omega}[W, \partial W] : \omega \neq (0)$$

where

$$\gamma^i \in \mathbf{K}^{2i}(BU) : i = 0, 1, \ldots$$

are the **K**-theory Chern classes. In the s_ω notation for characteristic classes the **K**-theory characteristic numbers of (U, fr)-manifolds are

$$s_\omega(\gamma)[W, \partial W] : \omega \neq \varnothing.$$

Observe finally that for a *closed* U-manifold C

$$s_\varnothing(\gamma)[C] = \mathfrak{I}[C] = Td[C],$$

the Todd genus of C. However for a general (U, fr)-manifold $(W, \partial W)$, $s_\varnothing(\gamma)[W, \partial W]$ is not defined and $Td[W, \partial W]$ is not a **K**-theory characteristic number of $(W, \partial W)$.

Our objective in this section is to show how relations among the $s_\omega(\gamma)$ numbers, $\omega \neq \varnothing$, of a closed U-manifold imply relations on its Todd genus. This we then apply to (U, fr) manifolds. We begin by recalling the Hattori–Stong theorem [8], [13].

The Hattori–Stong Theorem

Let us continue to denote by **BU** the spectrum representing $\mathbb{Z}$-graded **K**-theory. There is then a natural map

$$H : \mathbf{MU} \to \mathbf{BU} \wedge \mathbf{MU}$$

obtained by smashing with the unit of the ring spectrum **BU**. There thus arises a mapping

$$H_* : \pi_*(\mathbf{MU}) \to \pi_*(\mathbf{BU} \wedge \mathbf{MU})$$

by applying homotopy. Now recalling that

$$\pi_*(\mathbf{MU}) = \mathbf{\Omega}_*^U$$

$$\pi_*(\mathbf{MU} \wedge \mathbf{BU}) = \mathbf{K}_*(\mathbf{MU})$$

we may view H_* as a mapping

$$H_* : \mathbf{\Omega}_*^U \to \mathbf{K}_*(\mathbf{MU}).$$

The Hattori–Stong theorem may be formulated as follows:

THEOREM (Hattori-Stong). *The natural mapping*

$$H_* : \mathbf{\Omega}_*^U \to \mathbf{K}_*(\mathbf{MU})$$

is a split monomorphism of graded abelian groups.

We are now prepared to establish:

PROPOSITION 1.1. *Suppose that* $[M] \in \mathbf{\Omega}_{2n}^U$ *and* $n \not\equiv 0 \bmod p - 1$. *Assume that* p *is an odd prime and*

$$s_\omega(\gamma)[M] \equiv 0 \bmod p : \qquad \omega \neq \varnothing.$$

Then

$$Td[M] \equiv 0 \bmod p$$

and hence

$$[M] = p[N]$$

for some $[N] \in \mathbf{\Omega}_{2n}^U$.

Proof. Let $a = g.c.d.(n, p-1)$. Then $n = ab$, $p - l = ac$ for suitable $b, c \in \mathbb{Z}$. Thus

$$nc = abc = b(p-1).$$

Hence

$$\dim[M]^c = 2abc = \dim[CP(p-1)]^b.$$

Now recall [14] that

$$s_\omega(\gamma)[CP(p-1)] \equiv 0 \bmod p; \qquad \omega \neq \varnothing.$$

We shall need the following elementary lemma that arises from inductively staring at the formula

$$s_\omega(\gamma)[N' \times N''] = \sum_{\omega'\omega'' = \omega} s_{\omega'}(\gamma)[N']s_{\omega''}(\gamma)[N''].$$

LEMMA If $[N] \in \mathbf{\Omega}_*^U$ satisfies

$$s_\omega(\gamma)[N] \equiv 0 \bmod p: \qquad \omega \neq \varnothing$$

then the same is true of $[N]^s$ for any $s > 0$.**

Therefore

$$s_\omega(\gamma)([M]^c - Td[M]^c[\mathbf{C}P(p-1)]^b) \equiv 0 \bmod p; \qquad \omega \neq \varnothing.$$

But observe that by construction

$$Td([M]^c - Td[M]^c[\mathbf{C}P(p-1)]^b) = 0.$$

Thus

$$s_\omega(\gamma)([M]^c - Td[M]^c[\mathbf{C}P(p-1)]^b) \equiv 0 \bmod \mathrm{p}; \qquad \text{all } \omega.$$

That is all the **K**-theory characteristic numbers of

$$[M]^c - Td[M]^c[\mathbf{C}P(p-1)]^b$$

are congruent to zero mod p. Hence by the Hattori–Stong theorem

$$(*) \qquad [M]^c - Td[M]^c[\mathbf{C}P(p-1)]^b = p[C]$$

for some closed U-manifold C.

Pass to the quotient ring $\mathbf{\Omega}_*^U/(p)$, which is a graded polynomial ring over $\mathbb{Z}_p$ with one generator in each even degree. Recall that $[\mathbf{C}P(p-1)]$ may be taken to be the generator in dimension $2p-2$. Equation (*) takes on the form

$$(*p) \qquad [M]^c = Td[M]^c[\mathbf{C}P(p-1)]^b.$$

Now suppose that $Td[M]^c \not\equiv 0 \bmod p$. Then unique factorization in the graded polynomial ring $\mathbf{\Omega}_*^U/(p)$ implies in view of (*p) that

$$[M] = q[\mathbf{C}P(p-1)]^s$$

for some $q \neq 0 \in \mathbb{Z}_p$ and $s \in \mathbb{Z}$. But this means

$$2n = \dim M = 2s(p-1)$$

and hence

$$n \equiv 0 \bmod p-1$$

contrary to assumption. Therefore we must conclude that

$$Td[M]^c \equiv 0 \bmod p$$

and hence of course

$$Td[M] \equiv 0 \bmod p$$

as desired.

An application of the Hattori–Stong theorem now completes the proof. □

COROLLARY 1.2 *Suppose that p is an odd prime and that n is a positive integer with $n \not\equiv 0 \bmod p-1$. Then the mapping*

$$\Phi_*^{(p)} : \mathbf{\Omega}_{2n}^U \otimes \mathbb{Z}_p \to \frac{\mathbf{\Omega}_{2n}^{U,fr}}{\text{Torsion}} \otimes \mathbb{Z}_p$$

induced by the forgetful homomorphism is an isomorphism and

$$u_*^{(p)}: \frac{\Omega_{2n}^{U,fr}}{\text{Torsion}} \otimes \mathbb{Z}_p \to \mathbf{K}_{2n}(\mathbf{MU/S}) \otimes \mathbb{Z}_p$$

is a monomorphism.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \Omega_{2n}^{U} \otimes \mathbb{Z}_p & \xrightarrow{\Phi_*(p)} & \dfrac{\Omega_{2n}^{U,fr}}{\text{Torsion}} \otimes \mathbb{Z}_p \\ {\scriptstyle H_*(p)}\downarrow & & \downarrow{\scriptstyle u_*(p)} \\ \mathbf{K}_{2n}(\mathbf{MU}) \otimes \mathbb{Z}_p & \xrightarrow{\Psi_*(p)} & \mathbf{K}_{2n}(\mathbf{MU/S}) \otimes \mathbb{Z}_p. \end{array}$$

It follows from (1.1) that $u_*^{(p)} \cdot \Phi_*^{(p)}$ is a monomorphism. Hence $\Phi_*^{(p)}$ is monic. Recall that

$$\Phi_*^{(0)}: \Omega_{2n}^{U} \otimes \mathbf{Q} \to \Omega_{2n}^{U,fr} \otimes \mathbf{Q}$$

is an ismorphism for positive n. Therefore

$$\dim_{\mathbb{Z}_p} \Omega_{2n}^{U} \otimes \mathbb{Z}_p = \dim_{\mathbb{Z}_p} \frac{\Omega_{2n}^{U,fr}}{\text{Torsion}} \otimes \mathbb{Z}_p.$$

Hence $\Phi_*^{(p)}$ must be an isomorphism. Therefore $u_*^{(p)}$ must be monic, competing the proof. □

§2. ON A (U, fr) CELL

In this section we will introduce an explicit (U, fr) structure on the cell (E^{2n}, S^{2n-1}) and compute all the K-theory characteristic numbers of the resulting (U, fr) manifold. This manifold and its $s_\omega(\gamma)$ numbers are crucial to the proof of our main result.

PROPOSITION 2.1. *Let n be a positive integer. Then there exists a (U, fr) cell (E^{2n}, S^{2n-1}) such that $\partial_*[E^{2n}, S^{2n-1}] \in \Omega_{2n-1} = \pi_{2n-1}^S$ generates the image of*

$$J_{\mathbf{C}}: \pi_{2n-1}(U) \to \pi_{2n-1}^S$$

and

$$c_n(\tau) = (n-1)!\, e^{2n} \in H^{2n}(E^{2n}, S^{2n-1}; \mathbb{Z})$$

where $e^{2n} \in H^{2n}(E^{2n}, S^{2n-1}; \mathbb{Z})$ is the orientation class.

Proof. Let us recall that

$$\mathbf{K}(E^{2n}, S^{2n-1}) \cong \mathbf{K}(S^{2n}) \cong \mathbb{Z}.$$

If $\sigma \in \mathbf{K}(S^{2n})$ is the usual generator then [5]

$$c_n(\sigma) = (n-1)]!\, \iota_{2n} \in H^{2n}(S^{2n}; \mathbb{Z})$$

where ι_{2n} is the corresponding usual generator of cohomology. Denote by $\mu \in \mathbf{K}(E^{2n}, S^{2n-1})$ the element corresponding to σ. Clearly we then have

$$c_n(\mu) = (n-1)!\, e^{2n}.$$

Since $\widetilde{KO}(E^{2n}) = 0$, we may choose μ as a stable tangent bundle for (E^{2n}, S^{2n-1}) providing it with the structure of a (U, fr) manifold.

Let ϕ denote the standard (stable) framing of the stable tangent bundle of S^{2n-1}. There is then a mapping

$$\theta : S^{2n-1} \to U(N)$$

such that the framing of $\mu | S^{2n-1}$ is given by

$$\psi : S^{2n-1} \to C^N \times S^{2n-1}$$

where

$$\psi(x) = (\theta(x)\phi(x), x) : x \in S^{2n-1}.$$

Via the Pontrjagin–Thom construction we therefore find

$$J_{\mathbf{C}}[\theta] = [S^{2n-1}, \mu | S^{2n-1}] \in \pi^S_{2n-1} = \Omega^{fr}_{2n-1}.$$

From the manner in which we chose μ it follows that

$$J_{\mathbf{C}}[\theta] = \partial_*[E^{2n}, S^{2n-1}]$$

is a generator of

$$\operatorname{Im}\{J_{\mathbf{C}} : \pi_{2n-1}(U) \to \pi^S_{2n-1}\}$$

as required. □

Recall [5] that

$$\pi_{2n-1}(U) \to \pi_{2n-1}(O) = \begin{cases} \mathbb{Z} \xrightarrow{2} \mathbb{Z} : n \equiv 0 \bmod 4 \\ \mathbb{Z} \xrightarrow{1} \mathbb{Z} : n \equiv 2 \bmod 4 \end{cases}$$

and

$$\pi_{2n-1}(O) \equiv 0, \qquad \mathbb{Z}_2 : n = 1, 3 \bmod 4.$$

Therefore $\partial_*[E^{2n}, S^{2n-1}]$ generates the image of J_O if $n \equiv 2 \bmod 4$ and is twice a generator if $n \equiv 0 \bmod 4$.

Notation. Henceforth we will write merely (E^{2n}, S^{2n-1}) for the (U, fr) cell with stable tangent bundle $\mu \in \mathbf{K}(E^{2n}, S^{2n-1})$ such that

$$c_n(\mu) = (n-1)!\, e^{2n}.$$

Our purposes require that we compute the **K**-theory characteristic numbers of (E^{2n}, S^{2n-1}). To this end let us note

$$s_\omega(\gamma)[E^{2n}, S^{2n-1}] = \langle chs_\omega(\gamma)[\sigma], \iota_{2n} \rangle$$

where σ, ι_{2n} are as above. For according to [14; Lemma p. 119]

$$s_\omega(\gamma)[E^{2n}, S^{2n-1}] = \langle chs_\omega(\gamma)T(E^{2n}), e^{2n} \rangle,$$

but since

$$T(E^{2n}) = 1$$

this becomes

$$s_\omega(\gamma)[E^{2n}, S^{2n-1}] = \langle chs_\omega(\gamma), e^{2n}\rangle$$
$$= \langle chs_\omega(\gamma)[\sigma], \iota_{2n}\rangle$$

by naturality of the Chern character. Thus it suffices to compute

$$chs_\omega(\gamma)[\sigma] \in H^*(S^{2n}; \mathbf{Q}).$$

Introduce the standard collapse map

$$\underbrace{S^2 \times S^2 \times \cdots \times S^2}_{n\text{-factors}} \xrightarrow[f]{} S^{2n}.$$

Observe that

$$f^*(\sigma_{2n}) = \sigma_2{}^1 \cdots \sigma_2{}^n.$$

(the exterior tensor product of the canonical generators of $\tilde{\mathbf{K}}(S^2)$ of the various factors).

PROPOSITION 2.2 For each positive integer i we have

$$s_i(\gamma)[E^{2n}, S^{2n-1}] = \sum_{j=0}^{i} (-1)^{i-j}\binom{i}{j} j^n.$$

Proof. Observe that the collapsing map

$$f: S^2 \times \cdots \times S^2 \to S^{2n}$$

is of degree 1 and so it suffices to compute

$$\langle chf^*s_i(\gamma)(\sigma), [S^2 \times \cdots \times S^2]\rangle$$

where $[S^2 \times \cdots \times S^2] \in H_{2n}(S^2 \times \cdots \times S^2; \mathbb{Z})$ is the fundamental homology class. By naturality

$$f^*s_i(\gamma)(\sigma) = s_i(\gamma)(f^*(\sigma))$$

and as we have seen

$$f^*(\sigma) = \sigma_2{}^1 \cdots \sigma_2{}^n.$$

Let $H \in \mathbf{K}(S^2)$ be the canonical line bundle. Then

$$\sigma_2 = H - 1.$$

Thus

$$f^*(\sigma) = (H^{(1)} - 1)(\cdots)(H^{(n)} - 1)$$

where $H^{(k)}$ denotes the pullback of H along the projection on the k^{th} S^2 factor. As $s_i(\gamma)$ is primitive

$$s_i(\gamma)((H^{(1)} - 1) \cdots (H^{(n)} - 1)) = s_i(\gamma)(H^{(1)} - 1) \cdots s_i(\gamma)(H^{(n)} - 1).$$

Now recall that for a line bundle L

$$\sum_{i=0}^{\infty} \frac{s_i(\gamma)(L)}{i!} t^i = e^{Lt}$$

and hence

$$(e^{Ht} - 1) = \sum_{i=0}^{\infty} \frac{s_i(\gamma)(H - 1)}{i!} t^i$$

Therefore

$$\sum \frac{s_i(\gamma)f^*(\sigma)}{\cdot} t^i = \prod_{j=1}^{n} (e^{H^{(j)}t} - 1)$$

and hence we find by direct computation that

$$s_i(\gamma) = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} (H^{(1)} \cdots H^{(n)})^j + \quad \begin{Bmatrix} \text{terms that evaluate to zero} \\ \text{on } [S^2 \times \cdots \times S^2]. \end{Bmatrix}$$

Since

$$\langle ch(H^{(1)} \cdots H^{(n)j}), [S^2 \times S^2 \times \cdots S^2] \rangle = j^n$$

we obtain

$$\langle ch s_i(\gamma) f^*(\sigma), [S^2 \times \cdots \times S^2] \rangle = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^n$$

from which the result follows as noted above. □

We are still left with the problem of evaluating the above sum, or at least its congruence class mod p. I am indebted to my wife Mi-Soo Bae Smith for this next computation.

PROPOSITION 2.3. *Let p be an odd prime and t a positive integer. Then*

$$s_i(\gamma)[E^{2t(p-1)}, S^{2t(p-1)-1}] \equiv \begin{cases} (-1)^{i+1} \bmod p & : \ \text{if } i < p \\ 0 \quad \bmod p & : \ \text{if } i \geq p. \end{cases}$$

Proof. We must recall [7] the following theorem of Fermat.

THEOREM OF FERMAT. *Let $a \in \mathbb{Z}$, $a \neq 0$ and p a prime with $p \nmid a$. Then* $a^{p-1} \equiv 1 \bmod p$.

Consider now the sum $\sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^{t(p-1)}$. If $p \nmid j$ then $p \nmid j^t$ and hence we obtain

$$j^{t(p-1)} \equiv \begin{cases} 1 \bmod p: & \text{if } p \nmid j \\ 0 \bmod p: & \text{if } p \mid j. \end{cases}$$

Thus

$$\sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^{t(p-1)} = \sum_{\substack{j=0 \\ p \nmid j}}^{i} (-1)^{i-j} \binom{i}{j} \bmod p.$$

Now note that

$$0 = (1-1)^i = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j}.$$

Therefore

$$\begin{aligned} \sum_{\substack{j=0 \\ p \nmid j}}^{i} (-1)^{i-j} \binom{i}{j} &= \sum_{\substack{j=0 \\ p \nmid j}}^{i} (-1)^{i-j} \binom{i}{j} - \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} \\ &= - \sum_{\substack{j=kp \\ k=0}}^{[i/p]} (-1)^{i-j} \binom{i}{j}. \end{aligned}$$

Now let us write

$$i = rp + q : 0 \leq q < p.$$

Recall that

$$\binom{rp+q}{kp} \equiv \binom{r}{k} \bmod p.$$

Thus we find

$$s_i(\gamma)[E^{2t(p-1)}, S^{2t(p-1)-1}] \equiv -\sum_{k=0}^{[i/p]} (-1)^{i-kp} \binom{i}{kp} \bmod p$$
$$= (-1)^{q+1} \sum_{k=0}^{r} (-1)^{rp-kp} \binom{i}{kp} \bmod p$$

(recall p is an odd prime so $(-1)^{rp-kp} = (-1)^{r-k}) = (-1)^{q+1}(1-1)^r$.

Therefore since

$$(1-1)^r \equiv \begin{cases} 1 & \text{if} \quad r = 0 \\ 0 & \text{if} \quad r > 0 \end{cases}$$

we obtain

$$s_i(\gamma)[E^{2t(p-1)}, S^{2t(p-1)-1}] \equiv \begin{cases} (-1)^{i+1} \bmod p & : \ \text{if } i < p \\ \quad 0 \quad \bmod p & : \ \text{if } i \geq p \end{cases}$$

as required. □

For values of $i > p$ we shall require a finer understanding of the numbers $s_i(\gamma)[E^{2n}, S^{2n-1}]$. To this end we introduce:

Definition. For each pair of non-negative integers (a, b) define

$$S(a, b) = \frac{1}{b!} \sum_{c=0}^{a} (-1)^{a-c} \binom{b}{c} c^a$$

The numbers $S(a, b)$ are known to satisfy the functional equation [11; p. 33/(7)], [9; p. 169/(4)]

$$S(a+1, b) = S(a, b-1) + bS(a, b).$$

They are called Stirling numbers of the second kind. Starting from the initial conditions

$$S(0, 0) = 1 \quad \text{and} \quad S(0, b) = 0 \quad \text{if} \quad b > 0,$$

which follow from the definitions, we may obtain these numbers step by step with the aid of the above equation. It follows that the Stirling numbers of the second kind are non-negative integers, and $S(a, b) > 0$ if $a \geq b$.

From this discussion we therefore obtain:

PROPOSITION 2.4. *Let i and n be positive integers. Then*

$$s_i(\gamma)[E^{2n}, S^{2n-1}] = i!\, S(n, i).$$

In particular

$$s_i(\gamma)[E^{2n}, S^{2n-1}] \equiv 0 \bmod i!$$

for all positive integers i. □

PROPOSITION 2.5. *Let p be an odd prime and n a positive integer. Then*

$$\gamma^i[E^{2n}, S^{2n-1}] \equiv 0 \bmod p \quad : \quad i > p.$$

Proof. Recall the formula of Newton

$$s_i(\gamma) = i\gamma^i + \text{decomposables}.$$

As products are zero in $\mathbf{K}(E^{2n}, S^{2n-1})$ we conclude

$$s_i(\gamma)[E^{2n}, S^{2n-1}] = i\gamma^i[E^{2n}, S^{2n-1}]$$

From (2.4) we therefore obtain

$$\gamma^i[E^{2n}, S^{2n-1}] = (i-1)!\, S(n, i).$$

Now for $i > p$

$$(i-1)! \equiv 0 \bmod p$$

and the result follows. □

The residue of $\gamma^p[E^{2n}, S^{2n-1}] \bmod p$ is intimately tied up with the arithmetic of Stirling numbers of the second kind. I am indebted to Professor L. Carlitz for a proof of the following fact:

PROPOSITION 2.6 *Let n be a positive integer and p a prime. Then*

$$S(n, p) = \begin{cases} (-1)^{n+1} \bmod p & : \text{ if } n-1 \equiv 0 \bmod p-1, \\ 0 \quad\quad \bmod p & : \text{ otherwise.} \end{cases}$$

In particular for an odd prime p

$$S(t(p-1), p) \equiv 0 \bmod p.$$

Proof. We have

$$\begin{aligned} S(n, p) &= \frac{1}{p!} \sum_{j=0}^{p} (-1)^{n-j} \binom{p}{j} j^n \\ &= \frac{1}{(p-1)!} \sum_{j=1}^{p} (-1)^{n-j} \binom{p-1}{j-1} j^{n-1} + \frac{1}{(p-1)!} p^{n-1} \\ &\equiv -\sum (-1)^{n-j} \binom{p-1}{j-1} j^{n-1} \bmod p \end{aligned}$$

by Wilson's [7] theorem.

Since (as easily follows by induction)

$$\binom{p-1}{j-1} = (-1)^{j-1} \bmod p$$

we get

$$S(n, p) = (-1)^n \sum_{j=1}^{p-1} j^{n-1} \bmod p.$$

But

$$\sum_{j=1}^{p-1} j^{n-1} \equiv \begin{cases} -1 \bmod p & : \text{ if } p-1 \mid n-1 \\ 0 \bmod p & : \text{ otherwise} \end{cases}$$

as required. □

From (2.4) and (2.5) we now obtain:

PROPOSITION 2.7. *Let p be an odd prime and t a positive integer. Then*

$$\gamma^i[E^{2t(p-1)}, S^{2t(p-1)-1}] \equiv 0 \bmod p : i \geq p.$$

Proof. It only remains to consider the case $i = p$. From (2.4) we obtain

$$\begin{aligned}\gamma^p[E^{2t(p-1)}, S^{2t(p-1)-1}] &= (p-1)!\, S(t(p-1), p) \\ &\equiv 0 \bmod p\end{aligned}$$

by (2.6). □

PROPOSITION 2.8. *Let p be an odd prime and t a positive integer. Then*

$$s_\omega(\gamma)[E^{2t(p-1)}, S^{2t(p-1)-1}] = \begin{cases} (-1)^{n(\omega)+1}\mu_\omega \bmod p & : \ n(\omega) < p, \\ 0 \qquad\qquad \bmod p & : \ n(\omega) \geq p \end{cases}$$

where $\mu_\omega \mid n(\omega)$.

Proof. According to the usual algorithm for expressing symmetric functions in terms of elementary symmetric functions we find that

$$s_\omega(\gamma) = \mu_\omega \gamma^{n(\omega)} + \text{decomposables},$$

where $\mu_\omega \in \mathbb{Z}$ is a divisor of $n(\omega)$. Hence

$$s_\omega(\gamma)[E^{2t(p-1)}, S^{2t(p-1)-1}] = \mu_\omega \gamma^{n(\omega)}[E^{2t(p-1)}, S^{2t(p-1)-1}]$$

and the result follows from (2.3) and (2.7). □

3. THE MAIN RESULTS

Let p be an odd prime. We are going to show that the mapping of the introduction

$$u_*^{(p)} : \frac{\mathbf{\Omega}_*^{U,fr}}{\text{Torsion}} \otimes \mathbb{Z}_p \to \mathbf{K}_*(\mathbf{MU/S}) \otimes \mathbb{Z}_p$$

is a monomorphism.

To this end we must employ Stong's description of generators for $\mathbf{\Omega}_*^U \otimes \mathbb{Z}_p$ [13] [14; p. 125]. We begin by recalling that

$$\mathbf{K}^*(BU) = \mathbb{Z}[[\gamma^1, \ldots, \gamma^n, \ldots]]$$

and

$$\mathbf{K}_*(BU) = \mathbb{Z}[\beta_1, \ldots, \beta_n, \ldots]$$

where

$$\gamma^i \in \mathbf{K}^{2i}(BU)$$

is the ith K-theory Chern class and

$$\beta_i \in \mathbf{K}_{2i}(BU)$$

is dual to $s_i(\gamma)$. Via the Thom isomorphism for $\mathbf{K}$-theory the Hattori–Stong mapping

$$H_* : \Omega_*^U \to \mathbf{K}_*(\mathbf{M}U) = \mathbf{K}_*(BU)$$

is given by

$$H_*[M] = \sum_{\omega} s_\omega(\gamma)[M]\beta_\omega$$

where the sum ranges over all ordered partitions ω. (Note that $s_\omega(\gamma)[M] = 0$ for $2n(\omega) > \dim M$ so the sum is finite.)

For each prime p we obtain the mapping

$$H_*^{(p)} : \Omega_*^U \otimes \mathbb{Z}_p \to \mathbf{K}_*(\mathbf{M}U) \otimes \mathbb{Z}_p$$

for which Stong gives a convenient description. Following the account of [14] let us introduce:

Definition. If $P \in \mathbb{Z}_p[\beta_1, \ldots]$ we say that P has largest monomial $\beta_{i_1} \cdots \beta_{i_r}$ iff

(1) the coefficient of $\beta_{i_1} \cdots \beta_{i_r}$ in P is non-zero,

(2) if the coefficient of $\beta_{j_1} \cdots \beta_{j_s}$ in P is non-zero then either

(a) $j_1 + \cdots + j_s < i_1 + \cdots + i_r$, or

(b) $j_1 + \cdots + j_s = i_1 + \cdots + i_r$ and $s > r$.

Note that a polynomial need not have a largest monomial. If $P, Q \in \mathbb{Z}_p[\beta_1, \ldots]$ have largest monomials β_ρ and β_σ respectively then PQ has largest monomial $\beta_{\rho \cup \sigma}$. If $\{P_i \mid P_i \in \mathbb{Z}_p[\beta_1, \ldots]\}$ are polynomials with distinct largest monomials then they are linearly independent over $\mathbb{Z}_p$.

THEOREM (STONG). *Let p be a prime. Then there exist almost complex manifolds M_p^{2i} of dimension $2i$, $i = 1, 2, \ldots$ such that $H_*^{(p)}[M_p^{2i}]$ has largest monomial*

$$\begin{array}{lll} \beta_i & : \textit{if} & i+1 \neq p^s, \quad s > 0 \\ (\beta_{p^{s-1}-1})^p & : \textit{if} & i+1 = p^s, \quad s > 0. \end{array}$$

For $i + 1 \not\equiv 0 \bmod p$ one may choose $M_p^{2i} = \mathbf{C}P(i)$. In addition one may choose $M_p^{2p-2} = \mathbf{C}P(P-1)$.

For a proof of this result see [13; §5] or [14; pp. 117–127].

Thus the monomials in the manifolds $[M_p^{2i}]$ of dimension $2n$ form a $\mathbb{Z}_p$ basis for $\Omega_{2n}^U \otimes \mathbb{Z}_p$.

Notations. For each ordered partition

$$\omega = (r_1, \ldots, r_s)$$

define the moment of ω by

$$m(\omega) = \sum_{i=1}^{s} i r_i.$$

Let

$$M_p = (M_p{}^2)^{r_1} \times \cdots \times (M_p{}^{2s})^{r_s},$$

and note that

$$\dim M_p = 2m(\omega).$$

Definition. Let p be a prime and t a positive integer. Define for each partition ω with

$$\omega \neq \underbrace{(0, \ldots, t, 0, \ldots)}_{p-1-\text{position}}$$

$$2m(\omega) = 2t(p-1)$$

a (U, fr) manifold $N_p{}^\omega$ by

$$N_p{}^\omega = M_p{}^\omega.$$

For the partition

$$t\Delta_{p-1} = (0, \ldots 0, t, 0, \ldots)$$

excluded above let

$$N^{t\Delta}_{p^{p-1}} = (E^{2t(p-1)}, S^{2t(p-1)-1}),$$

the (U, fr) cell discussed in the previous section.

Let us examine the map

$$u_* : \frac{\mathbf{\Omega}_*{}^{U,fr}}{\text{Torsion}} \to \mathbf{K}_*(\mathbf{MU/S})$$

in more detail. From the dual exact sequences

$$0 \to \mathbf{K}_*(\mathbf{S}) \to \mathbf{K}_*(\mathbf{MU}) \to \mathbf{K}_*(\mathbf{MU/S}) \to 0$$

$$0 \leftarrow \mathbf{K}^*(\mathbf{S}) \leftarrow \mathbf{K}^*(\mathbf{MU}) \leftarrow \mathbf{K}_*(\mathbf{MU/S}) \leftarrow 0$$

obtained from the cofibration

$$\mathbf{S} \to \mathbf{MU} \to \mathbf{MU/S}$$

we find that we may describe $\mathbf{K}_*(\mathbf{MU/S})$ via the isomorphism

$$\mathbf{K}_*(\mathbf{MU/S}) = \mathbb{Z}[\beta_1, \ldots]/\mathbb{Z} \cdot 1.$$

Thus the elements of $\mathbf{K}_*(\mathbf{MU/S})$ may be viewed as sums of monomials $\beta_\omega : \omega \neq \varnothing$. The mapping

$$u_* : \frac{\mathbf{\Omega}_*{}^{U,fr}}{\text{Torsion}} \to \mathbf{K}_*(\mathbf{MU/S})$$

may be described by the formula

$$u_*[N] = \sum_{\omega \neq \varnothing} s_\omega(\gamma)[N]\beta_\omega .$$

As the elements of $\mathbf{K}_*(\mathbf{MU/S})$ may be viewed as polynomials in $\beta_1, \ldots$ with zero constant term we may employ our ordering on monomials as in $\mathbf{K}_*(\mathbf{MU})$.

PROPOSITION 3.1. *Let p be an odd prime and t a positive integer. Then the largest monomial in $u_*^{(p)}[N_p^{t\Delta}p-1]$ is β_{p-1}.*

Proof. This follows from (2.8) and the definitions. □

PROPOSITION 3.2. *Let p be an odd prime and t a positive integer. Then the manifolds $\{N_p^{\omega} \mid m(\omega) = t(p-1)\}$ have distinct largest monomials.*

Proof. This follows by observing that the monomial β_{p-1} can never be the largest monomial of one of the manifolds $\{M_p^{\omega} \mid m(\omega) = t(p-1)\}$ and (3.1). □

COROLLARY 3.3. *Let p be an odd prime and t a positive integer. Then the mapping*

$$u_*^{(p)} : \frac{\Omega_{2t(p-1)}^{U,fr}}{\text{Torsion}} \otimes \mathbb{Z}_p \to \mathbf{K}_{2t(p-1)}(\mathbf{MU/S}) \otimes \mathbb{Z}_p$$

is a monomorphism and the manifolds

$$\{N_p^{\omega} \mid m(\omega) = t(p-1)\}$$

form a $\mathbb{Z}_p$ basis for

$$\frac{\Omega_{2t(p-1)}^{U,fr}}{\text{Torsion}} \otimes \mathbb{Z}_p .$$

Proof. This follows from (3.2) upon recalling that the rank of

$$\frac{\Omega_{2t(p-1)}^{U,fr}}{\text{Torsion}}$$

is exactly the number of ordered partitions ω with $m(\omega) = t(p-1)$. □

Combining Corollaries 2.2 and 3.3 we obtain our main result:

THEOREM 3.4. *Let p be an odd prime. Then the mapping*

$$u_*^{(p)} : \frac{\Omega_*^{U,fr}}{\text{Torsion}} \otimes \mathbb{Z}_p \to \mathbf{K}_*(\mathbf{MU/S}) \otimes \mathbb{Z}_p$$

is a monomorphism. □

The results stated in the introduction now follow from elementary abelian group arguments and the **K**-theory Thom isomorphism.

As another application of (3.2) we note:

COROLLARY 3.5. *Let p be an odd prime and t a positive integer. Then the mapping*

$$\Phi_*^{(p)} : \Omega_{2t(p-1)}^{U} \otimes \mathbb{Z}_p \to \frac{\Omega_{2t(p-1)}^{U,fr}}{\text{Torsion}} \otimes \mathbb{Z}_p$$

has kernel the 1-dimensional $\mathbb{Z}_p$ subspace spanned by $[CP(p-1)^t]$.

Hence under the forgetful homomorphism Φ_* we have

$$\Phi_*[CP(p-1)^t] = p[W] \in \frac{\Omega_{2t(p-1)}^{U,fr}}{\text{Torsion}}$$

for some (U, fr) manifold W. □

It remains to remark that the analogs of these results for the prime 2 are false. This may be seen as follows:

PROPOSITION 3.6. *We have*

$$s_\omega(\gamma)[\mathbf{C}P(1)^2] \equiv 0 \bmod 2 : \omega \neq \varnothing$$

but

$$\Phi_*([\mathbf{C}P(1)^2]) \neq 2[W] \in \frac{\Omega_4^{U,fr}}{\text{Torsion}}.$$

Hence the mapping

$$u_*^{(2)} : \frac{\Omega_4^{U,fr}}{\text{Torsion}} \otimes \mathbb{Z}_2 \to \mathbf{K}_4(\mathbf{MU/S}) \otimes \mathbb{Z}_2$$

has a non-trivial kernel.

Proof. The computation of the $\mathbf{K}$-theory numbers of $[\mathbf{C}P(1)]^2$ may be found in [14; p. 120]. To see that $\Phi_*[\mathbf{C}P(1)^2]$ is not divisible by 2 in $\Omega_4^{U,fr}/\text{Torsion}$ we reason as follows. Suppose

$$\Phi_*[\mathbf{C}P(1)^2] = 2[W].$$

Then since $Td[\mathbf{C}P(1)^2] = 1$ we must have $Td[W] = 1/2$. Next recall [6; §6]

$$\frac{\Omega_4^{U,fr}}{\text{Torsion} + \Phi_*\Omega_4^U} \cong \mathbb{Z}_{24}$$

while [6; §15]

$$Td[E] = 1/12$$

where

$$[E] \in \frac{\Omega_4^{U,fr}}{\text{Torsion} + \Phi_*\Omega_4^U}$$

is a generator. As

$$[W] \in \frac{\Omega_4^{U,fr}}{\text{Torsion} + \Phi_*\Omega_4^U}$$

has order 2 we have

$$[W] = 12[E] + [C]$$

for some closed manifold C. Therefore $Td[W] \in \mathbb{Z}$ which is a contradiction. Hence $\Phi_*[\mathbf{C}P(1)^2]$ is not divisible by 2 and the result follows. □

It would be of technical interest to know what the z primary analog of our main results are. Is it for example true that

$$U_*^{(2)} : \frac{\Omega_n^{U,fr}}{\text{Torsion}} \otimes \mathbb{Z}_2 \to \mathbf{K}_n(\mathbf{MU/S}) \otimes \mathbb{Z}_2$$

is monic for $n \equiv z \bmod 4$? What is the situation for $n \equiv 0 \bmod 4$? (3.6) Shows that $U_*^{(2)}$ will not always be monic. What then is the index of $\Phi_*\left(\frac{\Omega_n^{U,fr}}{\text{Torsion}}\right)$ in $\mathbf{K}_n(\mathbf{MU/S})$?

§4. APPLICATIONS

The objective of this section is to show how the results of the previous section, in particular (3.2), may be applied to the study of the homomorphism

$$Td : \Omega_{2n}^{U,fr} \to \mathbf{Q}.$$

Due to the results of [6; §15] this study may be reinterpreted to determine the odd primary part of the image of the Adams e-invariant [3]

$$e_{\mathbf{C}} : \pi_{2n-1}^{S} \to \mathbf{Q}/\mathbb{Z}$$

and thereby rederive with the aid of [1; 3.7] the fact that the odd primary component of ImJ is a direct summand in $\pi_*{}^S$. We begin with:

PROPOSITION 4.1. *Let n be a positive integer and (E^{2n}, S^{2n-1}) the (U, fr) cell previously discussed with*

$$ch(\tau) = e^{2n} \in H^{2n}(E^{2n}, S^{2n-1}; \mathbf{Q}).$$

Then $(\Omega_{2n}{}^{U,fr}/\mathrm{Torsion}) \otimes \mathbb{Z}[1/2]$ is generated as a $\mathbb{Z}[1/2]$-module by

$$Im\left\{\Phi_*{}^{(1/2)} : \Omega_{2n}{}^{U} \otimes \mathbb{Z}[1/2] \to \frac{\Omega_{2n}{}^{U,fr}}{\mathrm{Torsion}} \otimes \mathbb{Z}[1/2]\right\}$$

and the element

$$[E^{2n}, S^{2n-1}] \in \Omega_{2n}{}^{U,fr}.$$

Proof. This follows directly from (3.2). □

Notations. (1) Continue to denote by (E^{2n}, S^{2n-1}) the (U, fr) cell discussed previously with

$$ch(\tau) = e^{2n} \in H^{2n}(E^{2n}, S^{2n-1}; \mathbf{Q}).$$

(2) Let B_n denote the nth Bernoulli number. More precisely [7; p. 90] let

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \beta_n \frac{x^n}{n!}.$$

The classical Bernoulli numbers may be defined by

$$B_n = (-1)^{n-1} \beta_{2n}$$

(Note that $\beta_{2m+1} = 0$, $m > 0$.)

PROPOSITION 4.2. *Let n be a positive integer. Then*

$$Td[E^{4n}, S^{4n-1}] = (-1)^{n-1} \frac{B_n}{2n}$$

$$Td[E^{4n+2}, S^{4n+1}] = 0.$$

Proof. As is well known*

$$Td_{4n} = \frac{(-1)^{n-1} B_n}{(2n-1)!\, 2n} c_n + \text{decomposables}$$

* See for example the discussion on pp. 11–14 of F. HIRZEBRUCH's book *Topological Methods in Algebraic Geometry*, Third Edition, Springer-Verlag, Berlin/New York (1966).

and

$$Td_{4n+2} = c_1 P_{4n+2}.$$

Hence we obtain:

$$\begin{aligned} Td[E^{4n}, S^{4n-1}] &= \frac{(-1)^{n-1} B_n}{(2n-1)!\,2n} \langle c_n(\tau), e^{2n} \rangle \\ &= \frac{(-1)^{n-1} B_n}{(2n-1)!\,2n}(2n-1)! = \frac{(-1)^{n-1} B_n}{2n} \end{aligned}$$

while

$$Td[E^{4n+2}, S^{4n+1}] = 0$$

since Td_{4n+2} is decomposable and products are zero in $H^*(E^{4n+2}, S^{4n+1}; \mathbf{Q})$. □

THEOREM 4.3. *Let n be a positive integer. Then the image of the homomorphism*

$$Td : \frac{\Omega_{4n}^{U,fr}}{\text{Torsion}} \otimes \mathbb{Z}[1/2] \to \mathbf{Q}$$

is the $\mathbb{Z}[1/2]$ submodule of $\mathbf{Q}$ generated by $(-1)^{n-1}B_n/2n$. The image of the homomorphism

$$Td : \frac{\Omega_{4n+2}^{U,fr}}{\text{Torsion}} \otimes \mathbb{Z}[1/2] \to \mathbf{Q}$$

is zero.

Proof. Recall that

$$Td[M] \in \mathbb{Z} : [M] \in \Omega_{2m}{}^{U}.$$

The result is immediate from (4.1) and (4.2). □

Remark. The above result constitutes merely the odd primary part of [6; 16.3]. However in [6] the results of Adams [3] on the homomorphism

$$\mathbf{e}_{\mathbf{C}} : \pi_{2m-1}^S \to \mathbf{Q}/\mathbb{Z}$$

are employed in the proof, while we are rederiving the odd primary part of Adams' results from (3.2) via (4.3).

THEOREM 4.4. *Let n be a positive integer. Then the image of the homomorphism*

$$\mathbf{e}_{\mathbf{C}}{}^{\text{odd}} : \pi_{4n-1}^S \to \mathbf{Q}/\mathbb{Z}[1/2]$$

consists of the cyclic subgroup of $\mathbf{Q}/\mathbb{Z}[1/2]$ generated by $(-1)^{n-1}B_n/2n$. The image of the homomorphism

$$\mathbf{e}_{\mathbf{C}}{}^{\text{odd}} : \pi_{4n+1}^S \to \mathbf{Q}/\mathbb{Z}[1/2]$$

is zero.

If $\theta_{2m-1} \in \pi_{2m-1}(U)$ is the usual generator [5] then

$$\mathbf{e}_{\mathbf{C}}{}^{\text{odd}} J_{\mathbf{C}}(\Theta_{2m-1}) = \begin{Bmatrix} \dfrac{(-1)^{n-1}B_n}{2n} & : & m = 2n \\ 0 & : & m \text{ odd} \end{Bmatrix} \in \mathbf{Q}/\mathbb{Z}[1/2].$$

Proof. According to Conner and Floyd [6; 16.2] the diagram

$$\begin{tikzcd} \Omega_{2m}^{U,fr} \arrow[r] \arrow[d, "Td"] & \Omega_{2m-1}^{fr} \arrow[r] \arrow[d, "e_C"] & 0 \\ \mathbb{Q} \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

is commutative. The result now follows from (4.3) and (2.1) upon recalling that

$$Td[M] \in \mathbb{Z} : [M] \in \Omega_{2m}^{U}.$$

and applying elementary abelian group theoretic considerations. □

Notations and definitions. (1) For any finitely generated abelian group A let A^{odd} denote the subgroup of A consisting of elements of odd order.

(2) Let n be a positive integer. Define an *odd* integer s_n by requiring

$$\text{Denominator}\left(\frac{B_n}{2n}\right) = 2^r s_n .$$

THEOREM 4.5. *Let n be a positive integer. Then the Todd genus homomorphism induces isomorphisms*

$$\left[\frac{\Omega_{4n}^{U,fr}}{\text{Torsion} + \Phi_*\Omega_{4n}^{U}}\right]^{\mathrm{odd}} \cong \mathbb{Z}_{s_n}$$

and

$$\left[\frac{\Omega_{4n+2}^{U,fr}}{\text{Torsion} + \Phi_*\Omega_{4n+2}^{U}}\right]^{\mathrm{odd}} \cong 0.$$

Proof. This is immediate from (4.3) and elementary group theoretic considerations. □

5. CLOSING REMARKS

It is perhaps instructive to apply the preceding discussions to make some concrete computations. We will therefore examine the cokernel of the Hurewicz map

$$\Omega_*^{U,fr} \to H_*(\mathbf{MU/S}; \mathbb{Z}).$$

Let us recall that according to Cohen [14; p. 130] that the cokernel of the Hurewicz map

$$\Omega_{2n}^{U} \to H_{2n}(\mathbf{MU/S}; \mathbb{Z})$$

is the direct sum of cyclic groups $\mathbb{Z}_{m_\omega}$ where ω is a partition of n and

$$m_i = \begin{cases} p & \text{if } i+1 = p^s \quad \text{for some prime } p; \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote this cokernel by C_{2n} and by D_{2n} let us denote the cokernel of

$$\Omega_{2n}^{U,fr} \to H_{2n}(\mathbf{MU/S}; \mathbb{Z}).$$

We may introduce the commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & A_{2n} \arrow[d] & \\
0 \arrow[r] & \Omega_{2n}{}^{U} \arrow[r] \arrow[d] & H_{2n}(\mathbf{MU};\, \mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & C_{2n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_{2n}{}^{U,fr} \arrow[r] \arrow[d] & H_{2n}(\mathbf{MU/S}\,;\, \mathbb{Z}) \arrow[r] & D_{2n} \arrow[r] \arrow[d] & 0 \\
 & E_{2n} \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

of exact sequences, defining the group E_{2n}. In view of (4.1), (4.2) and [6; 15.1] we obtain

$$E_{2n} \quad = \begin{cases} \mathbb{Z}/\dfrac{(-1)^{m-1}B^m}{2m}\,\mathbb{Z} & : \quad n = 2m \\ \qquad 0 & : \quad \text{otherwise.} \end{cases}$$

Therefore we find:

PROPOSITION 5.1. *Let the notations be as above and let n be a positive integer. Then there is an exact sequence*

$$0 \to E_{2n} \to C_{2n} \to D_{2n} \to 0$$

where

$$E_{2n}{}^{\text{odd}} = \begin{cases} \mathbb{Z}_{B_m/2m} & : \quad n = 2m \\ \ \ 0 & : \quad n \text{ odd.} \end{cases}$$

Proof. The result follows by applying the serpent lemma to the preceding diagram and the computation of the odd primary part of $E_{2n}{}^{\text{odd}}$ discussed above. □

From the preceding discussion we may readily compute the order of the odd primary part of the cokernel of

$$\Omega_{2n}{}^{U,fr} \to H_{2n}(\mathbf{M/US};\, \mathbb{Z}).$$

In particular we note that if n is an odd positive integer then the natural map

$$H_*(\mathbf{MU};\, \mathbb{Z}[1/2]) \to H_*(\mathbf{MU/S};\, \mathbb{Z}[1/2])$$

maps the image of

$$\Omega_{2n}{}^{U}/\text{Torsion} \otimes \mathbb{Z}[1/2] \to H_{2n}(\mathbf{MU};\, \mathbb{Z}[1/2])$$

isomorphically onto the image of

$$\Omega_{2n}{}^{U,fr}/\text{Torsion} \otimes \mathbb{Z}[1/2] \to H_{2n}(\mathbf{MU/S};\, \mathbb{Z}[1/2]).$$

Acknowledgement—During the preparation of this work I profited from several conversations with P. E. Conner, E. E. Floyd and R. E. Stong. I am also greatly indebted to my wife Mi-Soo Bae Smith for considerable help with the number theory that culminated in our realization of the role played by Stirling numbers in the theory.

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